Derivation of the Maxwell equations and the relation between electric and magnetic charge

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December 9, 1999

Abstract

We give a derivation of the complete set of the Maxwell equations based entirely on the Lagrangian derivability of the Newtonian equation of motion for a test particle and the self adjoint character of the differential operator. In the process, we are led to a fundamental relation between electric and magnetic monopole charges and thereby establishing that there essentially exists only one kind of charge which is by convention called electric.

PACS: 03.50.De, 45.20.Dd, 03.30.+p, 14.80.Hv

1 Introduction

In recent times, derivation of the Maxwell equations has attracted considerable attention. It was triggered off by Dyson's elegant paper [1] which discussed Feynman's derivation of the homogeneous Maxwell equations. It used commutation relations between coordinates and velocities rather than

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canonical momenta. It was soon realized that the problem is related to existence of an *Action/Lagrangian* for a given equation of motion. Since then a lot of effort has gone into building a relativistic generalization of Feynman's proof and its extension to non-Abelian gauge theories and to curved space [2, 3, 4]. In a recent paper [5] it has been shown that it is possible to incorporate magnetic monopoles in Feynman's formalism and hence derive the complete set of generalized Maxwell equations. However, the proof of generalized Maxwell equations involves some subtleties which remain unanswered. One of these is the proper identification of evolution parameter which makes interpretation rather unclear.

It may be noted that all these attempts involved commutation relations and quantum theory considerations. The question arises, could there not exist a derivation of the Maxwell equations based entirely in classical physics? This is the question we wish to address in this paper and we would give a derivation of the complete set of the Maxwell equations which only involves classical mechanics.

We begin by demanding that the Newtonian equation of motion for a test particle is derivable from a Lagrangian and the second order differential operator is self adjoint. This determines the form of the force, as that of the Lorentz force, involving a polar and an axial vector. These vector fields satisfy the homogeneous set of the Maxwell equations. However they are arbitrary and have nothing at this stage to do with the Maxwell electric and magnetic fields. By introducing scalar and pseudo-scalar, we further resolve the each vector into polar and axial vectors. We now have a set of four vectors, two each of polar and axial kinds, and two of a scalar and pseudo-scalar. Substituting these in the equation of motion and the two homogeneous equations, we get a set of four equations involving four vectors. It turns out that this set of equations, which is Maxwell-like but not actually the Maxwell equations, is invariant under the Galilean transformation. The covariance of the equation of motion determines the transformation laws for the fields occurring in the equation.

Note that for complete determination of a vector field, two equations giving divergence and curl are required. However the system of equations we have arrived at so far has only four equations for four vector fields, and hence is under determined. This system can only be solved by assuming linear relationship between the two polar and the two axial fields. We are thus led to assume the linear relations and contracting the system back to two vector fields, what we began with. With this, not only the set becomes the exact Maxwell set (of course we go from the Galilean to the Lorentz invariance) but the most remarkable relation that emerges from consistency of the equations is a relation between scalar and pseudo-scalar charges. That is between electric and magnetic monopoles. This leads to a very profound statement that there is no essential difference between electric and magnetic monopoles and one is a mirror image of the other. We thus need to consider only one of them and could by convention call it electric or magnetic.

One may wonder, what we have really done? We began with two fields which were provided by the self-adjoint and the Lagrangian derivable equation of motion. We then further split them into four and then recombined and miraculously got the Maxwell equations together with the wonderful synthesis of electric and magnetic charges and the Lorentz invariance. This is what we have really done. Why should this lead to such a deep and profound synthesis, we do not fully comprehend?

For quantization of electric charge, one resorts to the Dirac magnetic monopole. Unfortunately theories containing magnetic monopole could not be derived from an action and nor could the monopole fit well in the quantum electrodynamics. Hence, even its theoretical existence still remains an open question. In our formulation of electrodynamics, there is no room for any other monopole charge and hence quantization will have to come from some quantum consideration. It turns out that if we use the fine structure constant in our relation for scalar and pseudo-scalar charges, then quantization of charge emerges with a proper identification of parameter. Our consideration has been purely classical and hence it is not expected to give quantization relation unless we borrow something from the quantum regime.

The paper is organized as follows. In the next Sec., we briefly recall the discussion of self adjointness of the second order differential operator and the inverse problem in classical mechanics, which lead to the Lorentz-like force with the homogeneous set of two equations. In Sec. III, we derive the intermediate set which is Galilean invariant followed by in Sec. IV the derivation of the entire set of Maxwell equations and the fundamental relation between electric and magnetic charges. We conclude with a discussion of general issues and the ones to be taken up in future.

2 Self adjointness and the inverse problem

The inverse problem in classical mechanics deals with the demand of a Lagrangian for a given equation of motion. It turns out that if the equation of motion is self adjoint then the necessary and sufficient conditions for existence of a Lagrangian [3, 6] are

$$\frac{\partial \mathcal{F}_i}{\partial \ddot{q}^j} = \frac{\partial \mathcal{F}_j}{\partial \ddot{q}^i} \tag{1}$$

$$\frac{\partial \mathcal{F}_i}{\partial \dot{q}^j} + \frac{\partial \mathcal{F}_j}{\partial \dot{q}^i} = \frac{d}{dt}(\frac{\partial \mathcal{F}_i}{\partial \ddot{q}^j} + \frac{\partial \mathcal{F}_j}{\partial \ddot{q}^i}) \tag{2}$$

$$\frac{\partial \mathcal{F}_i}{\partial q^j} - \frac{\partial \mathcal{F}_j}{\partial q^i} = \frac{1}{2}\frac{d}{dt}(\frac{\partial \mathcal{F}_i}{\partial \dot{q}^j} - \frac{\partial \mathcal{F}_j}{\partial \dot{q}^i}) \tag{3}$$

where $\mathcal{F}_i$ is a system of second order differential equations,

$$\mathcal{F}_i(t, q, \dot{q}, \ddot{q}) = 0 \qquad i = 1, 2, ..., n. \tag{4}$$

Eqns (1 - 3) are known as the Helmholtz conditions and for the Newtonian equation of motion, we write $\mathcal{F}_i$ as

$$m\ddot{q}_i - F_i(t, q, \dot{q}) = 0 \tag{5}$$

where F_i is the force experienced by a test particle. Substitution of eqn (5) in eqns (1 - 3) yield

$$\frac{\partial F_i}{\partial \dot{q}^j} + \frac{\partial F_j}{\partial \dot{q}^i} = 0 \tag{6}$$

$$\frac{\partial^2 F_i}{\partial \dot{q}^j \partial \dot{q}^k} - \frac{\partial^2 F_j}{\partial \dot{q}^i \partial \dot{q}^k} = 0 \tag{7}$$

$$\frac{\partial F_i}{\partial q^j} - \frac{\partial F_j}{\partial q^i} = \frac{1}{2}(\frac{\partial}{\partial t} + \dot{q}^k \frac{\partial}{\partial q^k})[\frac{\partial F_i}{\partial \dot{q}^j} - \frac{\partial F_j}{\partial \dot{q}^i}]. \tag{8}$$

¿From eqns(6 & 7), it is easily seen that

$$m\ddot{q}_i = \lambda_i(t, q) + \xi_{ij}(t, q)\dot{q}^j \tag{9}$$

which when substituted in eqns(6 - 8) leads to

$$\begin{aligned}
\xi_{ij} + \xi_{ji} &= 0 \qquad (10)\\
\frac{\partial \xi_{ij}}{\partial q^k} + \frac{\partial \xi_{jk}}{\partial q^i} + \frac{\partial \xi_{ki}}{\partial q^j} &= 0 \qquad (11)\\
\frac{\partial \xi_{ij}}{\partial t} &= \frac{\partial \lambda_i}{\partial q^j} - \frac{\partial \lambda_j}{\partial q^i}. \qquad (12)
\end{aligned}$$

Eqns(9 - 12) are the necessary and sufficient conditions for existence of a Lagrangian for the Newtonian equation of motion. If we define

$$\lambda_i \equiv \mathcal{X}_i \qquad (13)$$

and

$$\xi_{ij} \equiv \epsilon_{ijk} \mathcal{Y}^k, \qquad (14)$$

then eqns(9, 11 & 12) can be written in the vector form as

$$\begin{aligned}
\vec{F} &= \vec{\mathcal{X}} + \vec{v} \times \vec{\mathcal{Y}} \qquad (15)\\
\vec{\nabla} \times \vec{\mathcal{X}} &= -\frac{\partial \vec{\mathcal{Y}}}{\partial t} \qquad (16)\\
\vec{\nabla} . \vec{\mathcal{Y}} &= 0. \qquad (17)
\end{aligned}$$

It is important to note here that $\vec{\mathcal{X}}$ and $\vec{\mathcal{Y}}$ are any arbitrary fields experienced by a test particle and eqns (15 - 17) will hold for *any* Newtonian force which has self adjoint equation of motion. These are the equations which were derived by Feynman in 1948 by assuming the commutation relation between coordinates and velocities rather than coordinates and canonically conjugate momenta. These equations can also be obtained by assuming the similar Poisson bracket relations [4, 7].

The derivation of the above form of the force (15) and the two homogeneous equations (16 & 17) hold good if and only if the second order differential operator in the equation of motion is self adjoint. Thus the demand of existence of a Lagrangian for a self adjoint equation of motion determines the form of the force. For a non self adjoint equation of motion, there does not exist a Lagrangian and nor do the eqns (15 - 17) [6]. This is the case whenever dissipative forces are involved and then the differential equation is not self adjoint. It could however be made self adjoint by introducing appropriate Lagrange multipliers [8]. It is well known that dissipative systems yield imaginary solutions which are attributed to the presence of sources and sinks. If these sources and sinks are properly included into the dynamical system under study, then the non self adjointness and consequently imaginary solutions disappear. That is self adjointness of the equation of motion guarantees real solutions while non self adjointness reflects ‘incompleteness’ of the system under consideration admitting imaginary solutions. In quantum mechanics, self adjoint or hermitian character of the operator guarantees that all the eigenvalues are real. Regardless of classical and quantum mechanics, self adjointness of the equation of motion and its derivability from a Lagrangian determines the character of force and the homogeneous equations (15-17) and reality of solutions/eigenvalues.

3 The Galilean invariant intermediate set of equations

In eqns (15-17), we have the Lorentz force and the homogeneous set of the Maxwell equations for the two fields involved. For derivation of the complete set of the Maxwell equations, we need only to bring the remaining two equations. This we shall do by first splitting the two vector fields into four and then recombining them. We note that $\vec{F}$ is a polar vector and so is $\vec{\mathcal{X}}$ while $\vec{\mathcal{Y}}$ is axial. We further decompose the vectors $\vec{\mathcal{X}}$ and $\vec{\mathcal{Y}}$ in terms of two polar $\vec{\mathcal{E}}$ & $\vec{\mathcal{D}}$ and two axial $\vec{\mathcal{B}}$ & $\vec{\mathcal{H}}$ vector fields as follows.

$$\vec{\mathcal{X}} = q_s\vec{\mathcal{E}} + q_p\vec{\mathcal{H}} \tag{18}$$

$$\vec{\mathcal{Y}} = q_s\vec{\mathcal{B}} - q_p\vec{\mathcal{D}} \tag{19}$$

where q_s indicates a constant scalar charge and q_p, the constant pseudo-scalar charge.

Substituting them in eqns(15-17), we obtain

$$
\begin{aligned}
\vec{F} &= q_s(\vec{\mathcal{E}} + \vec{v} \times \vec{\mathcal{B}}) + q_p(\vec{\mathcal{H}} - \vec{v} \times \vec{\mathcal{D}}) && (20) \\
\vec{\nabla} \times \vec{\mathcal{E}} &= -\frac{\partial \vec{\mathcal{B}}}{\partial t} && (21) \\
\vec{\nabla}.\vec{\mathcal{B}} &= 0 && (22) \\
\vec{\nabla} \times \vec{\mathcal{H}} &= \frac{\partial \vec{\mathcal{D}}}{\partial t} && (23) \\
\vec{\nabla}.\vec{\mathcal{D}} &= 0. && (24)
\end{aligned}
$$

This is the intermediate set which is Maxwellian like but not quite as it involves four independent vector fields. It can be easily checked that this set is invariant under the Galilean transformation because

$$
\begin{aligned}
\vec{\nabla}' &= \vec{\nabla} && (25) \\
\frac{\partial}{\partial t'} &= \frac{\partial}{\partial t} + \vec{V}.\vec{\nabla} && (26)
\end{aligned}
$$

The covariance of the force law (15) determines the following laws of transformations for the vector fields involved.

$$
\begin{aligned}
\vec{\mathcal{E}}' &= \vec{\mathcal{E}} + \vec{V} \times \vec{\mathcal{B}} && (27) \\
\vec{\mathcal{B}}' &= \vec{\mathcal{B}} && (28) \\
\vec{\mathcal{H}}' &= \vec{\mathcal{H}} - \vec{V} \times \vec{\mathcal{D}} && (29) \\
\vec{\mathcal{D}}' &= \vec{\mathcal{D}}. && (30)
\end{aligned}
$$

4 The Maxwell equations and the fundamental relation

Clearly we can not proceed further from the intermediate set (21-24) because it is under determined, four differential relations for four vector fields. In fact twice as many would be required for the system to be solvable. For determining a vector field both its divergence and curl must be given. Thus we are led to assume relations between the two polar and two axial vectors and that we do through a one more intermediate step as follows:

$$\vec{\mathcal{D}} = \epsilon\vec{\mathcal{E}}^* \quad (31)$$
$$\vec{\mathcal{B}}^* = \mu\vec{\mathcal{H}} \quad (32)$$

where $\vec{\mathcal{E}}^*$ & $\vec{\mathcal{B}}^*$ have the same dimensions of $\vec{\mathcal{E}}$ & $\vec{\mathcal{B}}$ respectively. Substituting the above relations in the intermediate set (20-24), we obtain

$$\vec{F} = q_s(\vec{\mathcal{E}} + \vec{v}\times\vec{\mathcal{B}}) + \frac{q_p}{\mu}(\vec{\mathcal{B}}^* - \mu\epsilon\vec{v}\times\vec{\mathcal{E}}^*) \quad (33)$$
$$\vec{\nabla}\times\vec{\mathcal{E}} = -\frac{\partial\vec{\mathcal{B}}}{\partial t} \quad (34)$$
$$\vec{\nabla}.\vec{\mathcal{B}} = 0 \quad (35)$$
$$\vec{\nabla}\times\vec{\mathcal{B}}^* = \mu\epsilon\frac{\partial\vec{\mathcal{E}}^*}{\partial t} \quad (36)$$
$$\vec{\nabla}.\vec{\mathcal{E}}^* = 0. \quad (37)$$

Although, $\vec{\mathcal{E}}$ & $\vec{\mathcal{E}}^*$ and similarly $\vec{\mathcal{B}}$ & $\vec{\mathcal{B}}^*$ have the same dimensions, their physical origin is very different. Since, we are working in a source free region, we can always assume the existence of sources of fields outside the region of our present consideration, and we make the following identifications,

$$e \equiv q_s \quad (38)$$
$$g \equiv \frac{q_p}{\mu}. \quad (39)$$

Now, if we identify $\vec{\mathcal{E}}$ as a field produced by a charge density ρ_e and $\vec{\mathcal{B}}^*$ as a field produced by a charge density ρ_g , then $\vec{\mathcal{E}}^*$ is the field produced by motion of ρ_g relative to test charge 'g' and is not produced by ρ_e. Similarly, $\vec{\mathcal{B}}$ is not produced by ρ_g but by the motion of ρ_e relative to test charge 'e'. Further symmetry of force law implies that one can not distinguish between the force in which either of the set $\vec{\mathcal{E}}$, $\vec{\mathcal{B}}$ or $\vec{\mathcal{E}}^*$, $\vec{\mathcal{B}}^*$ is absent. If both are present, then for physics to be consistent $\vec{\mathcal{E}}$ should be related to $\vec{\mathcal{E}}^*$ and $\vec{\mathcal{B}}$ should be related to $\vec{\mathcal{B}}^*$. This is anyway required for solvability of the system as argued earlier. For the overall consistency of the entire system of equations including the force law we are led to a profound conclusion that **there exists a fundamental relationship between the scalar charge** e **and the pseudo-scalar charge** g. We propose this relation to be

$$g = (\mu\epsilon)^{-1/2} e\, tan\theta \tag{40}$$

where θ is an invariant angle for a given family of particles in the universe. Similar relation has been considered earlier by Schwinger[9] for removing one kind of charge. This relation converts scalar into pseudo-scalar and vice-versa. Also, the set of eqns(34 - 37) is incomplete in the sense that we need to know both the divergence and the curl of a vector field to determine it completely. This does not happens until and unless $\vec{\mathcal{E}}$ & $\vec{\mathcal{E}}^*$ and similarly $\vec{\mathcal{B}}^*$ & $\vec{\mathcal{B}}$ differ only by a numerical factor.

Once the relationship between e and g is established, and the numerical factor taken care of in θ, one can replace $\vec{\mathcal{E}}^*$ by $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}^*$ by $\vec{\mathcal{B}}$ in eqn(33) to write force law in terms of charge 'e' as

$$\vec{F} = e(\vec{\mathcal{E}} + \vec{v} \times \vec{\mathcal{B}}) + (\mu\epsilon)^{-1/2} e\, tan\theta(\vec{\mathcal{B}} - \mu\epsilon\vec{v} \times \vec{\mathcal{E}}) \tag{41}$$

or

$$\vec{F} = e(\vec{\mathcal{E}} + (\mu\epsilon)^{-1/2}\, tan\theta\, \vec{\mathcal{B}}) + e(\vec{v} \times (\vec{\mathcal{B}} - (\mu\epsilon)^{1/2}\, tan\theta\, \vec{\mathcal{E}})). \tag{42}$$

We now define two fields $\vec{E}$ & $\vec{B}$ such that

$$\vec{E} = \vec{\mathcal{E}} + (\mu\epsilon)^{-1/2}\, tan\theta\, \vec{\mathcal{B}} \tag{43}$$

$$\vec{B} = \vec{\mathcal{B}} - (\mu\epsilon)^{1/2}\, tan\theta\, \vec{\mathcal{E}} \tag{44}$$

and hence

$$\vec{F} = e(\vec{E} + \vec{v} \times \vec{B}). \tag{45}$$

Inversely, one can rewrite eqns(43 & 44) as

$$\vec{\mathcal{E}} = cos^2\theta\, \vec{E} - (\mu\epsilon)^{-1/2}\, cos\theta\, sin\theta\, \vec{B} \tag{46}$$

$$\vec{\mathcal{B}} = cos^2\theta\, \vec{B} + (\mu\epsilon)^{1/2}\, cos\theta\, sin\theta\, \vec{E}. \tag{47}$$

Substituting these identifications of $\vec{\mathcal{E}}$ & $\vec{\mathcal{B}}$ in eqns(34 & 35), we obtain

$$cos^2\theta\, (\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t}) = cos\theta\, sin\theta\, ((\mu\epsilon)^{-1/2}\vec{\nabla} \times \vec{B} - (\mu\epsilon)^{1/2}\frac{\partial \vec{E}}{\partial t}) \tag{48}$$

$$cos^2\theta\, \vec{\nabla}.\vec{B} = -(\mu\epsilon)^{1/2}\, cos\theta\, sin\theta\, \vec{\nabla}.\vec{E}. \tag{49}$$

For these equations to hold independent of the value of θ we must have

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{50}$$

$$\vec{\nabla}.\vec{B} = 0 \tag{51}$$

$$\vec{\nabla} \times \vec{B} = \mu\epsilon \frac{\partial \vec{E}}{\partial t} \tag{52}$$

$$\vec{\nabla}.\vec{E} = 0 \tag{53}$$

which is the complete Maxwell set for the electric and magnetic fields $\vec{E}$ and $\vec{B}$. (These equations can also be obtained by substituting eqns(46 & 47) in eqns(36 & 37), with $\vec{\mathcal{E}}^*$ & $\vec{\mathcal{B}}^*$ replaced by $\vec{\mathcal{E}}$ & $\vec{\mathcal{B}}$ respectively).

It should however be noted that the Lorentz force law we have obtained could have equally been written down in terms of the pseudo-scalar magnetic charge g,

$$\vec{F} = g(\vec{B} - \mu\epsilon\vec{v} \times \vec{E}) \tag{54}$$

and the physics would have remained unchanged. That is, once e and g are related through eqn (40), it is only a matter of convention how does one write the force law and identifies 'electric' and 'magnetic' fields. This relation means that if electric and magnetic charges exist, then they must be related and ultimately there is only one independent charge, call it electric or magnetic.

Obviously the Maxwell equations are invariant under the Lorentz transformation and the Lorentz force is covariant defining the familiar relativistic transformation laws for electric and magnetic fields. One might wonder, how does the Lorentz invariance gets automatically woven in, particularly in view of the Galilean invariance of the intermediate set? True, so long as there was no relation between the two polar and two axial vectors, the transformation for invariance was Galilean. It turns to the Lorentz when the linear relations like(31-32), which throw up an invariant speed, are assumed between the two pairs of polar and axial fields. It is this that makes the difference and in fact leads to the Special Relativity (SR) aided with the equivalence of all inertial frames which follows from Newton's First Law. Here is a purely classical mechanics based path to SR in the sense that the Maxwell equations, which give invariance of speed of light, are themselves derived from the classical mechanics considerations.

If we write $c = (\mu\epsilon)^{-1/2}$, the speed of light, eqn (40) will read as

$$g = e\, c\, tan\theta. \tag{55}$$

Finally, by taking into account the regions containing sources and currents, and using the continuity equation

$$\frac{\partial \rho_e}{\partial t} + \vec{\nabla}.\vec{J_e} = 0 \tag{56}$$

where

$$\begin{aligned} \rho_e &= \rho_g\, cot\theta && (57) \\ \vec{J_e} &= \vec{J_g}\, cot\theta. && (58) \end{aligned}$$

The Maxwell equations in the source occupied region would be given by

$$\begin{aligned} \vec{\nabla}.\vec{E} &= \frac{\rho_e}{\epsilon} && (59) \\ \vec{\nabla}.\vec{B} &= 0 && (60) \\ \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} && (61) \\ \vec{\nabla} \times \vec{B} &= \mu\epsilon \frac{\partial \vec{E}}{\partial t} + \mu \vec{J_e}. && (62) \end{aligned}$$

This completes the derivation of the Maxwell equations and the fundamental relation connecting electric and magnetic monopole charges.

5 Discussion

We have essentially begun with the half of the Maxwell equations. We then split the two vector fields into four to write the intermediate set, which is Maxwell - like as it involves four rather than two fields. Now when we set the linear relationship between the two pairs of polar and axial fields, the intermediate set ultimately reduces to the complete set of Maxwell equations with the fundamental relation between the two kinds of monopole charges. It is therefore not very surprising that our imaginative exercise of splitting and recombining leads to the Maxwell equations, however what is unexpected is the fundamental relation. This has actually emerged as a byproduct. This is undoubtedly very profound and the most interesting result of the paper.

Magnetic monopole was first introduced by Dirac [10, 11] and it was envisioned as one end of an infinite string of dipoles or a solenoid. It did a wonderful job of quantizing electric charge even if one such entity existed in the whole Universe. This was quite remarkable but the problem with monopoles is that a theory containing them cannot be derived from an action principle [12, 13]. Further they led to singularity problems related to strings [14] and nor do they fit well in the quantum electrodynamics [15, 16]. Thus magnetic monopole has become an enigma, for it is required for the charge quantization but it could not successfully be accommodated in the existing theories.

In our formulation, there is however no room for the two charges to exist independently. One is simply the mirror image of the other. Of course the question of quantization remains. For that we have to appeal to some quantum principle or relation. By using the Dirac quantization condition and the fine structure constant, it is straight forward to write our fundamental relation (55). In CGS units the Dirac quantization condition is

$$\frac{eg}{\hbar c} = \frac{n}{2} \tag{63}$$

where n is an integer. Using this in the fine structure constant relation

$$\alpha = \frac{e^2}{\hbar c} \tag{64}$$

we get

$$g = \frac{n}{2\alpha}\, e. \tag{65}$$

Defining

$$tan\theta = \frac{n}{2\alpha} \tag{66}$$

leads to our relation in CGS units

$$g = e\, tan\theta. \tag{67}$$

Conversely, let us begin with the above relation and write it as $eg = e^2 tan\theta$, divide both sides by $\hbar c$ and choose $e^2 tan\theta/\hbar c = n/2$ to obtain the Dirac quantization condition (63). This is a way of getting at the charge quantization but we are not fully happy with it. This is one of the questions that will engage us in future. The other important questions to be addressed would include consideration of the fundamental relation in the context of the early Universe when the electromagnetic field is unified with the other fields and generalization of the formalism to internal degrees of freedom, curved space and general relativity (GR). Most importantly, could we construct a derivation of GR on similar lines. On the face of it, it looks rather difficult and unlikely. However we do believe that some ingenious and imaginative extrapolations may lead to something worth while.

We thank S. Mukherjee for helpful discussions and J. V. Narlikar for reading the manuscript.

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